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ON A SYSTEM OF HYPOCYCLOIDS OF CLASS THREE INSCRIBED TO A GIVEN 3-LINE, AND SOME CURVES CONNECTED WITH IT

BY H. A. CONVERSE

Introduction. In 1857 Steiner[*] stated the theorem: if from any point t on the circumcircle of a given triangle, $\alpha\beta\gamma$, the perpendiculars be drawn to the three sides of the triangle, their feet lie on a line[†] L; and as the point t moves around the circle the envelope of the line L is a particular curve of class three (and order four). He also gave some of the properties of this curve.

Later, in 1865, Cremona[‡] showed this curve (the envelope of the Simson-line of a given triangle) to be a hypocycloid of three cusps inscribed to the given triangle.

Poncelet[§] gave the theorem: if we project with constant angle of projection any point on the circumcircle of a given triangle on the three sides of the triangle, the points of projection lie on a line, which I shall call the *Poncelet-line* of the triangle. The Simson-line is a particular case of the Poncelet-line when the projection is orthogonal.

I shall generalize these results of Steiner and Cremona as follows: For any constant angle of projection, θ, the envelope of the Poncelet-lines of a given triangle is a hypocycloid of three cusps inscribed to the triangle; as the angle of projection varies we obtain a system of hypocycloids of three cusps inscribed to the triangle.

It is this system of hypocycloids, and some curves connected with it that I shall discuss in the following paper. First, I shall give some explanation of the system of coordinates which I shall use. Secondly, I shall make the extension, referred to above, of the Steiner-Cremona problem. Thirdly, I shall discuss a problem proposed by Professor Allardice, relating to this system of hypocycloids. Fourthly, I shall examine the cubic arising as the cusp-locus of the system of hypocycloids, and show the application of Study's *theory of osculants* to this method of handling a cubic. And to this I shall add a treatment of a second problem of Professor Allardice's by this system of coordinates.

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[*] J. Steiner: Über eine besondere Curve dritter Klasse (und vierten Grades), *Crelle*, vol. 53, p. 231 (1857).

[†] This line is known as the *Simson line*. See J. Allison: On the so-called Simson line, *Proceedings Edinburgh Math. Soc.*, vol. 3, p. 77 (1885); also *Nature*, vol. 30, p. 635 (1884).

[‡] M. L. Cremona, à Bologne: Sur l'hypocycloide à trois rebroussements, *Crelle*, vol. 64, p. 101 (1865).

[§] Poncelet: Propriétés projectives (1865), §468.

Following Professor Morley, I shall call a hypocycloid of three cusps a *deltoid.*[*]

1. The system of coordinates.

I shall use a system of coordinates sometimes called *circular coordinates*[†] and sometimes called *conjugate coordinates.*[‡]

Let t be a turn, *i. e.* a complex quantity of absolute value unity. It may then be said that t remains always on a circle of radius unity in the complex plane and any curve may be considered as the *map* of the unit circle by means of one equation, *e. g.*,

$$x = a_1 + 2a_2t + a_3t^2.$$

With this equation is connected its conjugate:

$$y = b_1 + 2\,\frac{b_2}{t} + \frac{b_3}{t^2},$$

where b_j is the conjugate of a_j.

Then $x = X + iY$ and $y = X - iY$ are called the *circular* or *conjugate* coordinates of the point x, given by any particular value of t; X and Y being the Cartesian coordinates of the point. As t runs around the unit circle, x runs along the curve. If t be eliminated from the map-equation of a curve and its conjugate, the resulting equation $\phi(x, y) = 0$, will be the conjugate equation of the curve. *E. g.*, $x = t$ is the map-equation of the unit circle; eliminating t from this and its conjugate, $y = 1/t$, we obtain $xy = 1$, the conjugate equation of the unit circle. A point may then be considered as given by one coordinate x (or its conjugate y) in the complex plane, or by two coordinates (x, y) in the Cartesian plane, given by the transformation

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[*] F. Morley: Orthocentric properties of the plane n-line, *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 1.

[†] F. Franklin: Some applications of circular coordinates, *Amer. Jour. Math.*, vol. 12 (1890), p. 161.

[‡] F. Morley: On the metric geometry of the plane n-line, *Trans. Amer. Math. Soc.*, vol. 1 (1900), pp. 97-115.

$$(1) \qquad \begin{cases} x = X + iY, \\ y = X - iY, \end{cases}$$

which makes the new axes of coordinates the circular rays.

Any equation of the first degree in x and t gives a circle; thus, $x = a_1 + a_2 t$ is the map-equation of the circle C in figure 1.

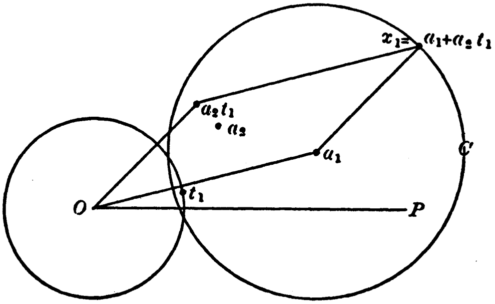

FIG. 1.

For, consider the quantities as vectors and the points as their extremities. Let a_1 and a_2 be any two complex quantities and t_1 any point on the unit circle. Let OP be the axis of reals. The point $x_1 = a_1 + a_2 t_1$ is the sum of the strokes a_1 and $a_2 t_1$. Since t is a turn the point $a_2 t$ will lie on a circle with centre at O and radius $|a_2|$; hence x_1 lies on a circle with centre at a_1 and radius $|a_2|$.

Any equation of the first degree in x and y, of the form $Ax + By + 1 = 0$, represents a straight line if A and B are conjugates, or, what is the same thing, if the equation is its own conjugate. This is readily seen if we apply the inverse of the transformation (1) to any equation of the first degree in X and Y.

The angle, θ, between two lines:

$$x + ay = \text{constant},$$

and

$$x + kay = \text{constant},$$

is given by the equation $e^{2i\theta} = k$. (See Franklin's paper referred to above.) By means of the transformation (1) it is evident that the envelope of a system of curves, given in terms of a parameter, is obtained by differentiating with regard to the parameter and eliminating the parameter.

The equation of a deltoid may be obtained as follows. A deltoid is traced by a point x, on the circumference of a circle C, of radius r, which rolls inside a circle C', of radius $3r$. Let the line OK (figure 2) be the axis of reals,

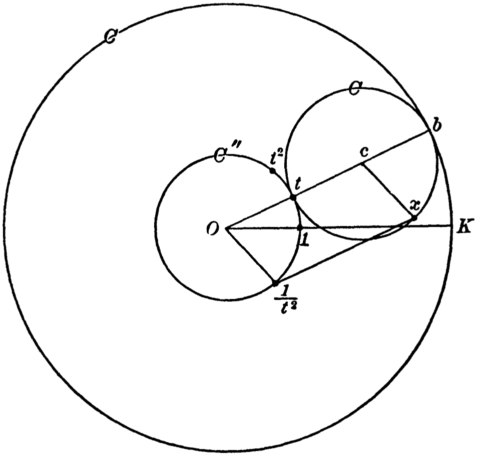

FIG. 2.

and let $r = cx = 1$. Then C'' is the unit circle. The arc $bK =$ arc $bx = 3$ arc $t1$ $=$ arc $t\,\frac{1}{t^2}$. Hence if we take the quantities as vectors, we have

$$x = 2t + \frac{1}{t^2}. \tag{2}$$

If the radius of C be $|r|$, (2) becomes

$$x = 2rt + \frac{r}{t^2}. \tag{3}$$

If the origin be transferred to any point $-a$, (3) becomes

$$x = a + 2rt + \frac{r}{t^2}. \tag{4}$$

This is the general map-equation of a deltoid with centre at a, and *size* $|r|$, where by the *size* of a deltoid we mean the radius of the inscribed circle.

Hence we may say in general that a deltoid is given by an equation of the form

$$x = a_1 + a_2 t + \frac{a_3}{t^2}, \text{ if} \left| \frac{a_2}{a_3} \right| = 2.$$

2. The system of deltoids. The extension of the Steiner-Cremona problem, mentioned above, is made as follows. Take $\alpha\beta\gamma$ as the given triangle, and let its circumcircle be the unit circle. The equation of the side $\beta\gamma$ is

$$\begin{vmatrix} \beta, & \frac{1}{\beta}, & 1 \\ \gamma, & \frac{1}{\gamma}, & 1 \\ x, & y, & 1 \end{vmatrix} = 0,$$

or

$$(1) \qquad x + \beta\gamma y = \beta + \gamma.$$

A line making a constant angle, θ, with (1) is given by the equation

$$kx + \beta\gamma y = \text{constant},$$

where

$$e^{2i\theta} = 1/k.$$

If this line cut the unit circle at a point t, its equation becomes

$$(2) \qquad kx + \beta\gamma y = kt + \frac{\beta\gamma}{t}.$$

Combining (1) and (2), we have their point of intersection given by

$$(3) \qquad \begin{cases} (k-1)\,x = kt + \dfrac{\sigma_3}{\alpha t} - \sigma_1 + \alpha, \\ -\dfrac{(k-1)}{k}\,y = \dfrac{1}{kt} + \dfrac{\alpha t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\alpha}, \end{cases}$$

where σ_1, σ_2, σ_3 are the symmetric functions of α, β, γ.

In like manner the points of intersection of the two other lines through t, making the angle θ with $\alpha\gamma$ and $\alpha\beta$ respectively, are given by

$$(3') \qquad \begin{cases} (k-1)x = kt + \dfrac{\sigma_3}{\beta t} - \sigma_1 + \beta, \\ -\dfrac{(k-1)}{k}\,y = \dfrac{1}{kt} + \dfrac{\beta t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\beta}, \end{cases}$$

and

$$(3'')\qquad \begin{cases} (k-1)\,x = kt + \dfrac{\sigma_3}{\gamma t} - \sigma_1 + \gamma, \\[2mm] -\dfrac{(k-1)}{k}\,y = \dfrac{1}{kt} + \dfrac{\gamma t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\gamma}. \end{cases}$$

The determinant of these three points is

$$\Delta \equiv \frac{-k}{(k-1)^2}\begin{vmatrix} kt + \dfrac{\sigma_3}{\alpha t} - \sigma_1 + \alpha, & \dfrac{1}{kt} + \dfrac{\alpha t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\alpha}, & 1 \\[2mm] kt + \dfrac{\sigma_3}{\beta t} - \sigma_1 + \beta, & \dfrac{1}{kt} + \dfrac{\beta t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\beta}, & 1 \\[2mm] kt + \dfrac{\sigma_3}{\gamma t} - \sigma_1 + \gamma, & \dfrac{1}{kt} + \dfrac{\gamma t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\gamma}, & 1 \end{vmatrix}.$$

On subtracting the second row from the first and the third from the second, and taking out the common factors, this becomes

$$\Delta \equiv \frac{-k(\beta-\alpha)(\gamma-\beta)}{t\sigma_3(k-1)^2}\begin{vmatrix} \gamma - t, & \gamma - t \\ \alpha - t, & \alpha - t \end{vmatrix} \equiv 0.$$

Hence the three points lie on a line, the Poncelet line of the triangle for the angle of projection θ and the point t. Its equation is

$$\frac{-k}{(k-1)^2}\begin{vmatrix} kt + \dfrac{\sigma_3}{\alpha t} - \sigma_1 + \alpha, & \dfrac{1}{kt} + \dfrac{\alpha t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\alpha}, & k-1 \\[2mm] kt + \dfrac{\sigma_3}{\beta t} - \sigma_1 + \beta, & \dfrac{1}{kt} + \dfrac{\beta t}{\sigma_3} - \dfrac{\sigma_2}{\sigma_3} + \dfrac{1}{\beta}, & k-1 \\[2mm] x \quad , & y \quad , & 1 \end{vmatrix} = 0,$$

or

$$(4)\qquad (k-1)tx + \frac{(k-1)\sigma_3}{k}\,y + \frac{\sigma_3}{kt} - \sigma_2 - kt^2 + \sigma_1 t = 0.$$

As t runs around the unit circle, θ and therefore k being constant, equation (4) gives us a system of lines. To obtain the envelope of this system, differentiate (4) with regard to t and obtain

$$(k-1)\,x = -\,\sigma_1 + 2kt + \frac{\sigma_3}{kt^2},$$

or

$$(5) \qquad x = \frac{\sigma_1}{1-k} - \frac{2kt}{1-k} - \frac{\sigma_3}{(1-k)kt^2}\,.$$

The result of eliminating t between (5) and (4) is easily seen to be the same as that of eliminating t between (5) and its conjugate. But (5) is the map-equation of a deltoid with its centre at the point $\frac{\sigma_1}{1-k}$ and its size $\left|\frac{-\,\sigma_3}{k(1-k)}\right|$. It touches the three sides of the given triangle; for, putting t equal to α/k, β/k, and γ/k in the equation (4), we obtain

$$x + \beta\gamma y = \beta + \gamma,$$
$$x + \gamma\alpha y = \gamma + \alpha,$$
$$x + \alpha\beta y = \alpha + \beta,$$

which are the three sides of the triangle.

Hence the envelope of the line (4) is a deltoid touching the three sides of the triangle. Now every possible value of k, for real values of θ, is in absolute value unity, since $1/k = e^{2i\theta}$. Hence as the angle θ takes all real values k is a turn, and for each value of k, (5) gives a deltoid. Hence, for a varying k, we have a system of deltoids, all inscribed to the given triangle. Their centres lie on a line; for the centre of (5) is given by

$$(6) \qquad x = \frac{\sigma_1}{1-k}\,.$$

As k varies (6) gives the locus of centres of the system. Eliminating k from (6) and its conjugate, $y = -\,\frac{\sigma_2 k}{\sigma_3(1-k)}$, we have the line

$$(7) \qquad \sigma_2 x + \sigma_1\sigma_3 y = \sigma_1\sigma_2.$$

This line is perpendicular to the *Euler-line* of the triangle, *i. e.* the line joining the circumcentre and the orthocentre of the triangle; for the equation of the Euler-line is

$$\begin{vmatrix} x, & y, & 1 \\ 0, & 0, & 1 \\ \sigma_1, & \dfrac{\sigma_2}{\sigma_3}, & 1 \end{vmatrix} = 0,$$

or

$$\sigma_2 x - \sigma_1 \sigma_3 y = 0,$$

and the condition of perpendicularity is satisfied.

The line (7) cuts the Euler-line at the point $\sigma_1/2$, the centre of the nine-point-, or Feuerbach-circle. That is, the centres of the deltoids of the system lie on the perpendicular bisector of the line joining the circumcentre to the orthocentre of the triangle, and hence are equidistant from these two points.

If $k = -1$, then $\theta = \pi/2$ and (5) becomes

$$(5') \qquad x = \frac{\sigma_1}{2} + t + \frac{\sigma_3}{2t^2}\cdot$$

This is Steiner's case, a deltoid with centre at $\sigma_1/2$, and size $\frac{1}{2}$.

The results so far obtained may be combined in the following theorem:

Theorem I. *If from a point t, on the circumcircle of a given triangle $\alpha\beta\gamma$, three lines be drawn making the angle θ with the three sides of the triangle respectively, their points of intersection with these sides lie on a line. The envelope of this line as t runs around the circle is a deltoid. As θ varies we obtain a system of deltoids, all inscribed to the given triangle and having their centres on a line perpendicular to the Euler-line of the triangle at a point midway between the orthocentre and the circumcentre of the triangle.*

Now combining (5) with its conjugate

$$-\frac{(1-k)}{k}\, y = \frac{\sigma_2}{\sigma_3} - \frac{2}{kt} - \frac{kt^2}{\sigma_3},$$

we obtain the involution of the curve

$$(8) \qquad t^3 + \left[\frac{1-k}{k}x - \frac{\sigma_1}{k}\right]t^2 + \left[\frac{1-k}{k}\sigma_3 y + \frac{\sigma_2}{k}\right]t - \frac{\sigma_3}{k^2} = 0\cdot$$

If the three roots of this equation fall together, *i. e.* if $t_1 = t_2 = t_3 = t$, then

$t^3 = \sigma_3/k^2$, and $t = \sqrt[3]{\sigma_3/k^2}$ gives the cusps of the deltoid. Putting this value of t in (5) we have the three cusps given by

$$x = \frac{\sigma_1 - 3\sqrt[3]{\sigma_3 k}}{1 - k}.$$

As k runs around the unit circle this equation gives the cusp-locus of the system of deltoids. And since σ_3 and k are both turns $\sqrt[3]{\sigma_3 k}$ is a turn, and we may write t for $\sqrt[3]{\sigma_3 k}$. Then the cusp-locus becomes

(9) $$x = \frac{\sigma_1 - 3t}{1 - \dfrac{t^3}{\sigma_3}}.$$

Hence:

THEOREM II. *The locus of the three cusps of the system of deltoids of Theorem I is a rational cubic curve whose map-equation is equation* (9).

Theorems I and II may be proved in another way by the use of certain notions introduced by Professor Morley in the paper referred to above (1903). Following this paper for the moment, let t_1, t_2, t_3 be three points on the unit circle, such that

(10) $$t_1 t_2 t_3 = 1.$$

Connect with these a point x by the equation

(11) $$x = t_1 + t_2 + t_3.$$

From (10) and (11) we have

(12) $$x = t_1 + t_2 + \frac{1}{t_1 t_2}.$$

In this let t_1 be fixed and t_2 variable. Since $x = t + 1/t$ is that simplest of all hypocycloids, the segment of a line, so (12) is the segment of a line of constant length 4. When t_1 varies the deltoid is described by the motion of this variable segment, which touches the curve and whose ends move on the curve. The curve itself is given when the segments are brought into coincidence, *i. e.* when

(13) $$x = 2t + \frac{1}{t^2}.$$

We may now prove Theorems I and II very simply.

The equation (1) as before is the chord $\beta\gamma$, and (2) is a line cutting (1) at a constant angle, and passing through a given point t of the unit circle; and

(3) gives the point of intersection of (1) and (2). Write the first equation of (3) in the form

$$(14)\qquad (k-1)x+\sigma_1=kt+\frac{\sigma_3}{at}+a\cdot$$

If the point given by (14) be on a segment of a line touching a deltoid, (14) must be identical with

$$(15)\qquad (k-1)x+\sigma_1=\lambda\left(t_1+t_2+\frac{1}{t_1t_2}\right);$$

therefore

$$kt=\lambda t_1,\qquad a=\lambda t_2,\qquad \frac{\sigma_3}{at}=\frac{\lambda}{t_1t_2},$$

whence we have

$$\lambda^3=\sigma_3k.$$

Putting this value of λ in (15) and making $t_1=t_2=t$ we have

$$(k-1)x+\sigma_1=2\sqrt[3]{\sigma_3k}\,t+\frac{\sqrt[3]{\sigma_3k}}{t^2},$$

or

$$(16)\qquad x=\frac{\sigma_1}{1-k}-\frac{2\sqrt[3]{\sigma_3k}}{1-k}-\frac{\sqrt[3]{\sigma_3k}}{(1-k)t^2},$$

a deltoid with centre at $\frac{\sigma_1}{1-k}$ and size $\frac{1}{|1-k|}$. This gives us the Theorem I.

For the cusps we have $t_1=t_2=\sqrt[3]{1}$. Then (15) becomes

$$(k-1)x+\sigma_1=3\lambda.$$

Eliminating k from this equation and $\lambda^3=\sigma_3k$, we have

$$(17)\qquad x=\frac{\sigma_1-3\lambda}{1-\dfrac{\lambda^3}{\sigma_3}},$$

which is, for variable λ, the cusp-locus of the system; hence the Theorem II.

Consider now any deltoid of the system (5), and compare it with the deltoid of Steiner's case (5′).

The size of (5′) is $\frac{1}{2}$, while that of (5) is $\frac{1}{|1-k|}$. The ratio of the size of (5) to that of (5′) is $\dfrac{2}{|1-k|}$. But $0 \leqq |1-k| \leqq 2$. Hence

$$\infty \geqq \frac{1}{|1-k|} \geqq \tfrac{1}{2}.$$

Hence the deltoid of Steiner's case is the minimum deltoid of the system and we have the

THEOREM III. *The minimum deltoid inscribed to a given triangle is the deltoid of Steiner's case, i. e. the envelope of the Simson-line of the triangle.*

Since deltoids are symmetrical figures, the angle which one deltoid makes with another may be taken as the angle which a chosen cusp tangent of one makes with the cusp tangents of the other.

The cusps of (5′) are given by $t^3 = \sigma_3$, and are

$$\begin{cases} x = \dfrac{\sigma_1}{2} + \dfrac{3\sqrt[3]{\sigma_3}}{2}, \\[2ex] y = \dfrac{\sigma_2}{2\sigma_3} + \dfrac{3}{2\sqrt[3]{\sigma_3}}. \end{cases}$$

The cusp tangents of (5′) are then given by

$$\begin{vmatrix} x\,, & y\,, & 1 \\[1ex] \dfrac{\sigma_1}{2} + \dfrac{3\sqrt[3]{\sigma_3}}{2}, & \dfrac{\sigma_2}{2\sigma_3} + \dfrac{3}{2\sqrt[3]{\sigma_3}}, & 1 \\[2ex] \dfrac{\sigma_1}{2}\,, & \dfrac{\sigma_2}{2\sigma_3}\,, & 1 \end{vmatrix} = 0,$$

or

$$x - \sqrt[3]{\sigma_3^2}\,y + \text{constant} = 0. \tag{18}$$

In like manner the cusps of (5) are given by $t^3 = \dfrac{\sigma_3}{k^2}$, and are

$$\begin{cases} x = \dfrac{\sigma_1}{1-k} - \dfrac{3\sqrt[3]{\sigma_3 k}}{1-k}, \\[2ex] y = \dfrac{-\sigma_2 k}{\sigma_3(1-k)} + \dfrac{3k}{(1-k)\sqrt[3]{\sigma_3 k}}. \end{cases}$$

Hence the cusp tangents of (5) are given by

$$\begin{vmatrix} x\ , & y\ , & 1 \\ \frac{\sigma_1}{1-k} - \frac{3\sqrt[3]{\sigma_3 k}}{1-k}, & \frac{-\sigma_2 k}{\sigma_3(1-k)} + \frac{3k}{(1-k)\sqrt[3]{\sigma_3 k}}, & 1 \\ \frac{\sigma_1}{1-k}, & \frac{-\sigma_2 k}{\sigma_3(1-k)}\ , & 1 \end{vmatrix} = 0,$$

or

$$(19) \qquad x + \sqrt[3]{\frac{\sigma_3^2}{k}}\, y + \text{constant} = 0.$$

Hence ϕ, the angle between (18) and (19), is given by

$$(20) \qquad e^{2i\phi} = -\sqrt[3]{\frac{1}{k}}\,.$$

Since $e^{2i\theta} = 1/k$, the formula connecting ϕ and θ is

$$(21) \qquad 3\phi - \theta = 3n\pi,$$

where n is any integer. Hence we have the

THEOREM IV. *Any deltoid of the system may be obtained by three transformations, from the minimum deltoid:—first, a translation perpendicular to the Euler-line of the triangle; second, a rotation about the centre; and third, a dilation.*

In other words, if $\mu = \frac{2}{|1-k|}$ be the magnification and ϕ the angle of rotation from the Steiner position, *a deltoid inscribed to a given triangle expands and rotates according to the law* $\mu = \frac{2}{|1+e^{-6i\phi}|}$, *keeping its centre on a fixed line, and its cusps on a definite cubic curve.*

3. Professor Allardice's problem. Professor Allardice* has proved that if we have a system of conics circumscribed to a given triangle, such that their asymptotes form a constant angle, the envelope of the asymptotes is two deltoids which are inscribed to the given triangle; and that as the angle between the asymptotes varies there is a single infinity of such pairs of deltoids. He states further that there is evidently some relation between the two deltoids of each pair, but leaves this undetermined. It appears at once that this system of deltoids is identical with that of section 2, and by the methods there used I shall easily determine the required relation. I shall first prove Professor Allardice's proposition by these methods, obtaining the equations of the pair of deltoids in a form such that their relation appears very simply.

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* R. E. Allardice: On some curves connected with a system of similar conics, ANNALS OF MATH. series 2, vol. 3, pp. 154–160 (1902).

Let there be two lines

$$\begin{cases} x - kay - b = 0, \\ x - \dfrac{a}{k} y - b' = 0, \end{cases} \tag{1}$$

the first making with the second an angle θ, given by the equation

$$e^{2i\theta} = k^2. \tag{2}$$

Take as before the given triangle to be $\alpha\beta\gamma$ and its circumcircle to be the unit circle. The equation of the circumcircle is then

$$xy = 1. \tag{3}$$

A conic whose asymptotes make the angle θ, is

$$x^2 - a\left(k + \frac{1}{k}\right)xy + a^2y^2 + dx + ey + f = 0. \tag{4}$$

Let this conic cut the unit circle in the three points α, β, γ, and a fourth point t. From (3) and (4) we obtain

$$x^4 + dx^3 + \left[f - a\left(k + \frac{1}{k}\right)\right]x^2 + ex + a^2 = 0, \tag{5}$$

an equation whose roots must be α, β, γ, and t. Hence (5) must be identical with

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0, \tag{6}$$

where S_1, S_2, S_3, S_4 are the symmetric functions of α, β, γ, t. Let σ_1, σ_2, σ_3 be the symmetric functions of α, β, γ and for simplicity take the Euler-line of the triangle as the axis of reals. Then σ_1 is real and hence equal to its conjugate σ_2/σ_3. And we may write for σ_2 its equal $\sigma_1\sigma_3$. Making these substitutions we then have

$$\begin{cases} S_1 = \sigma_1 + t, \\ S_2 = \sigma_1\sigma_3 + \sigma_1 t, \\ S_3 = \sigma_3 + \sigma_1\sigma_3 t, \\ S_4 = \sigma_3 t. \end{cases}$$

Making use of these equations in identifying (5) and (6) we have

$$
(7) \qquad \begin{cases} -\sigma_1 - t = d, \\ \sigma_1\sigma_3 + \sigma_1 t = f - a\left(k + \dfrac{1}{k}\right), \\ -\sigma_3 - \sigma_1\sigma_3 t = e, \\ \sigma_3 t = a^2. \end{cases}
$$

From the four equations (7) we readily determine the values of d, e, and f to be

$$
(8) \qquad \begin{cases} d = -\dfrac{\sigma_1\sigma_3 + a^2}{\sigma_3}, \\ e = -\sigma_3 - \sigma_1 a^2, \\ f = \sigma_1\sigma_3^2 + \sigma_1 a^2 + a\sigma_3\left(k + \dfrac{1}{k}\right). \end{cases}
$$

That the lines (1) shall be actual asymptotes of (4) it is necessary that

$$
(9) \qquad \begin{cases} b + b' = d, \\ \dfrac{b}{k} + kb' = \dfrac{e}{a}. \end{cases}
$$

Solving these equations for b and b' and substituting in the result the values of d and e from (8) we obtain

$$
(10) \qquad \begin{cases} b = \dfrac{(a^3k + \sigma_1\sigma_3 a^2 + \sigma_1\sigma_3 ka + \sigma_3^2)k}{a\sigma_3(k^2 - 1)}. \\ b' = \dfrac{a^3 + \sigma_1\sigma_3 ka^2 + \sigma_1\sigma_3 a + \sigma_3^2 k}{a\sigma_3(1 - k^2)}. \end{cases}
$$

Putting these values of b and b' in (1) we have the two asymptotes given in terms of a single parameter a in the forms

$$
(11) \qquad \frac{\sigma_3(k^2 - 1)}{a}x - \sigma_3 k(k^2 - 1)y - ak^2 - \sigma_1\sigma_3 k - \frac{\sigma_1\sigma_3 k^2}{a} - \frac{\sigma_3^2 k}{a^2} = 0,
$$

and

$$
(12) \qquad \frac{k\sigma_3(1 - k^2)}{a}x - \sigma_3(1 - k^2)y - ak - \sigma_1\sigma_3 k^2 - \frac{\sigma_1\sigma_3 k}{a} - \frac{\sigma_3^2 k^2}{a^2} = 0.
$$

To obtain the equations of their envelopes, differentiate (11) and (12) with regard to a and we have

$$x = \frac{\sigma_1 k^2}{k^2 - 1} - \frac{k^2 a^2}{\sigma_3(k^2 - 1)} + \frac{2k\sigma_3}{(k^2 - 1)a},$$

and

$$x = \frac{\sigma_1}{1 - k^2} - \frac{a^2}{\sigma_3(1 - k^2)} + \frac{2k\sigma_3}{(1 - k^2)a}.$$

Replacing in these two equations $\frac{\sigma_3}{a}$, which is a turn, by t, we have

$$x = \frac{\sigma_1 k^2}{k^2 - 1} + \frac{2kt}{k^2 - 1} - \frac{k^2 \sigma_3}{(k^2 - 1)t^2}, \tag{13}$$

and

$$x = \frac{\sigma_1}{1 - k^2} + \frac{2kt}{1 - k^2} - \frac{\sigma_3}{(1 - k^2)t^2}. \tag{14}$$

From their form equations (13) and (14) are seen to be deltoids. That they are inscribed to the triangle $\alpha\beta\gamma$ is evident geometrically from the fact that a system of conics through three points contains pairs of lines, one of which passes through two of the points and the other through the third. Analytically this appears as follows. The line through two of the points β and γ is

$$x + \beta\gamma y = \beta + \gamma.$$

If this be a tangent to the deltoid (13) there must be some value of a which is a turn, which will make (11) identical with this. This value is found to be $-\beta\gamma/k$. Hence the deltoid (13) touches the line $\beta\gamma$. In like manner we can show that (13) touches the other two sides of the triangle and also that (14) touches the three sides of the triangle. Hence we have the

THEOREM V (Allardice). *The envelope of the asymptotes of a system of similar conics through three points is two deltoids each inscribed to the triangle of the three points.*

Equations (13) and (14) may be reduced to the form (5) of section 2, showing that both belong to the system of section 2 and we may say, *the system of deltoids inscribed to a given triangle breaks up into pairs.*

To obtain the relation between the two deltoids of a pair, we examine the equations (13) and (14). The centre of (13) is $\frac{\sigma_1 k^2}{k^2 - 1}$; that of (14) is $\frac{\sigma_1}{1 - k^2}$. These two points are conjugates, hence the centre of (13) is the reflection of the centre of (14) in the axis of reals, *i. e.* the Euler-line.

The size of (13) is $\left|\frac{-k^2\sigma_3}{k^2-1}\right| = \frac{1}{|k^2-1|}$, and that of (14) is $\left|\frac{-\sigma_3}{1-k^2}\right| = \frac{1}{|1-k^2|}$. These are equal, *i. e.*, the two deltoids are of the same size. If we call the distance from the centre of a deltoid to the origin δ, and its size σ, we have as the formula connecting δ and σ,

$$\delta = \sigma_1\sigma\,; \tag{15}$$

i. e. the size is directly proportional to the distance from the origin. This expression is independent of k and hence holds for all deltoids of the system. Hence we have the

Theorem VI. *The centre of one deltoid of a pair is the reflection, in the Euler-line of the triangle, of the centre of the other. The two deltoids of a pair are equal in size. The ratio of the distance of the centre of any deltoid of the system from the circumcentre of the triangle, to the size of that deltoid, is constant, being equal to the distance of the orthocentre from the circumcentre.*

If the asymptotes are perpendicular, equation (2) becomes $e^{i\pi} = k^2$, hence $k^2 = -1$, and the system of conics becomes a system of rectangular hyperbolas; equations (13) and (14) then become

$$x = \frac{\sigma_1}{2} \mp it - \frac{\sigma_3}{2t^2}, \tag{13'}$$

and

$$x = \frac{\sigma_1}{2} \pm it - \frac{\sigma_3}{2t^2}. \tag{14'}$$

These two equations give the same deltoid, *i. e.* when the asymptotes are perpendicular the two deltoids of the pair coincide, and we have the well known theorem that the envelope of the asymptotes of a system of rectangular hyperbolas circumscribed to a given triangle is the minimum deltoid inscribed to the triangle.

It remains to determine the angle between the two deltoids of a pair. Call it ϕ. In exactly the same manner as in section 2 we find the angle ϕ between (13) and (14) to be given by

$$e^{2i\phi} = \sqrt[3]{k^4}. \tag{16}$$

Eliminating k between this and equation (2) we have the relation between ϕ and θ in the form

$$3\phi - 2\theta = n\pi \qquad (n = \text{any integer}). \tag{17}$$

Hence the deltoid (13) may be made to take the position (14) by a reflection in the Euler-line of the triangle, together with a rotation about its centre through an angle ϕ. Hence we may modify the theorem of Professor Allardice as follows :

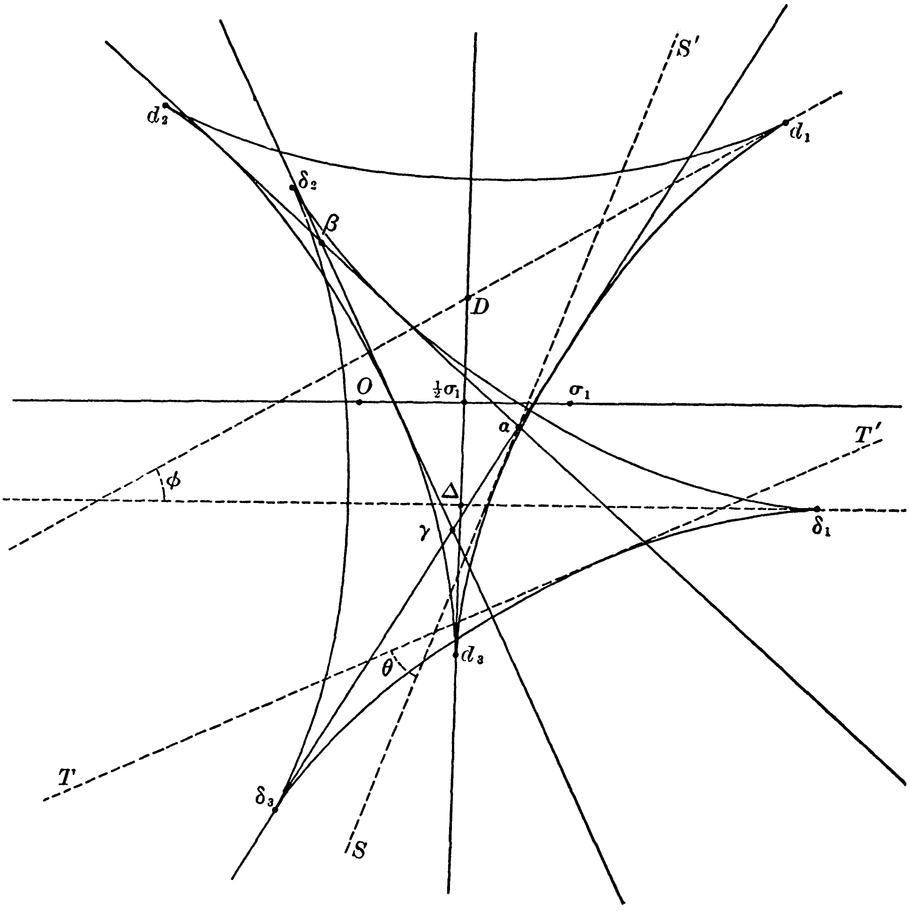

FIG. 3.

Every pair of deltoids of the same size determine a triangle, the triangle of their common tangents, such that these two deltoids shall be the envelope of the asymptotes of a definite system of similar conics circumscribed to that triangle.

Figure 3 shows two deltoids of a pair, both inscribed to the triangle $\alpha\beta\gamma$.

The circumcentre of the triangle is O, the orthocentre is σ_1, and the line $O\sigma_1$ is the Euler-line of the triangle. The point D, the centre of one deltoid, is seen to be the reflection, in $O\sigma_1$, of Δ, the centre of the other. SS' and TT' are a pair of asymptotes cutting at the angle θ, and Dd_1, and $\Delta\delta_1$, two cusp tangents cutting at the angle ϕ. The figure is drawn for the case where $\theta = \frac{\pi}{4}$ and hence $\phi = \frac{\pi}{6}$.

4. The cusp-locus of the system. Let us examine the cubic of Theorem II. Write its equation and that of its conjugate in the form

$$(1)\qquad \begin{cases} x = \dfrac{\sigma_1\sigma_3 - 3\sigma_3 t}{\sigma_3 - t^3}, \\[2ex] y = -\dfrac{\sigma_2 t^3 - 3\sigma_3 t^2}{\sigma_3(\sigma_3 - t^3)}. \end{cases}$$

Let the cubic be cut by any line

$$(2)\qquad Ax + By + 1 = 0.$$

Eliminating x and y from these three equations we obtain a cubic equation in t,

$$(3)\qquad t^3 - \frac{3B\sigma_3}{B\sigma_2 + \sigma_3}\, t^2 + \frac{3A\sigma_3^2}{B\sigma_2 + \sigma_3}\, t - \frac{A\sigma_2\sigma_3^2 + \sigma_3^2}{B\sigma_2 + \sigma_3} = 0,$$

whose three roots give the three points of intersection of the line with the cubic. To find the involution of points in which a line cuts the cubic, we let t_1, t_2, t_3 be the three roots of (3), and S_1, S_2, S_3 their symmetric functions. Equation (3) is then identical with

$$t^3 - S_1 t^2 + S_2 t - S_3 = 0,$$

and we have

$$(4)\qquad \begin{cases} S_1 = \dfrac{3B\sigma_3}{B\sigma_2 + \sigma_3}, \\[2ex] S_2 = \dfrac{3A\sigma_3^2}{B\sigma_2 + \sigma_3}, \\[2ex] S_3 = \dfrac{A\sigma_1\sigma_3^2 + \sigma_3^2}{B\sigma_2 + \sigma_3}, \end{cases}$$

or

$$(5)\qquad \begin{cases} (S_1\sigma_2 - 3\sigma_3)B + S_1\sigma_3 = 0, \\ 3\sigma_3^2 A - S_2\sigma_2 B - S_2\sigma_3 = 0, \\ \sigma_1\sigma_3^2 A - S_3\sigma_2 B - S_3\sigma_3 + \sigma_3^2 = 0. \end{cases}$$

The condition that these three equations hold for all values of A and B is the vanishing of the determinant

$$\begin{vmatrix} 0\ , & S_1\sigma_2 - 3\sigma_3, & S_1\sigma_3 \\ 3\sigma_3^2\ , & -S_2\sigma_2\ \ , & -S_2\sigma_3 \\ \sigma_1\sigma_3^2, & -S_3\sigma_2\ \ , & -S_3\sigma_3 + \sigma_3^2 \end{vmatrix},$$

or

$$(6)\qquad 3S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 = 0.$$

This is the involution of points in which a line cuts the cubic. If the three points coincide we have the triple points of the involution and the line cuts the cubic in three consecutive points. The line is then a flex-tangent of the cubic and its point of intersection with the cubic is a point of inflexion.

Let $t_1 = t_2 = t_3 = t$. Then $S_1 = 3t$, $S_2 = 3t^2$, $S_3 = t^3$, and equation (6) becomes

$$(7)\qquad t^3 - \sigma_1 t^2 + \sigma_2 t - \sigma_3 = 0.$$

The roots of (7) are α, β, γ. Hence the cubic has *three real points of inflexion* given by $t = \alpha$, $t = \beta$, $t = \gamma$. They are

$$(8)\qquad \begin{cases} x = \dfrac{(\sigma_1 - 3\alpha)\sigma_3}{\sigma_3 - \alpha^3}, \\ x = \dfrac{(\sigma_1 - 3\beta)\sigma_3}{\sigma_3 - \beta^3}, \\ x = \dfrac{(\sigma_1 - 3\gamma)\sigma_3}{\sigma_3 - \gamma^3}. \end{cases}$$

To determine the flex-tangent at the point given by $t = \alpha$, we put $t_1 = t_2 = t_3 = \alpha$, and we have $S_1 = 3\alpha$, $S_2 = 3\alpha^2$, $S_3 = \alpha^3$. Putting these values in equations (4) we have

$$A = \frac{-1}{\beta + \gamma}, \quad B = \frac{-\beta\gamma}{\beta + \gamma}.$$

Equation (2) then becomes

$$x + \beta\gamma y = \beta + \gamma,$$

the flex-tangent at the point given by $t = \alpha$. In the same way we find the other two flex-tangents. The three are

$$(9) \qquad \begin{cases} x + \beta\gamma y = \beta + \gamma, \\ x + \gamma\alpha y = \gamma + \alpha, \\ x + \alpha\beta y = \alpha + \beta. \end{cases}$$

But these are the three sides of the triangle. Hence the

THEOREM VII. *The cubic of Theorem* II *has three real points of inflexion given by the values of t corresponding to the vertices of the triangle, and the flex-tangent corresponding to each vertex is the opposite side, i. e. the points of inflexion of the cubic lie on the three sides of the triangle.*

To determine the line of flexes, put $t_1 = \alpha$, $t_2 = \beta$, $t_3 = \gamma$. Then $S_1 = \sigma_1$, $S_2 = \sigma_2$, $S_3 = \sigma_3$, and from equations (4) we obtain

$$A = \frac{\sigma_2}{3\sigma_3 - \sigma_1\sigma_2},$$

$$B = \frac{\sigma_1\sigma_3}{3\sigma_3 - \sigma_1\sigma_2}.$$

Putting these values in (2) we have

$$(10) \qquad \sigma_2 x + \sigma_1\sigma_3 y = \sigma_1\sigma_2 - 3\sigma_3,$$

the equation of the line of flexes. The equation of the Euler-line of the triangle (section 2, p. 112) is

$$(11) \qquad \sigma_2 x - \sigma_1\sigma_3 y = 0.$$

The lines (10) and (11) are perpendicular and meet at the point

$$(12) \qquad x = \frac{\sigma_1\sigma_2 - 3\sigma_3}{2\sigma_2}.$$

Hence the

THEOREM VIII. *The line of flexes of the cubic is perpendicular to the Euler-line of the triangle at the point*

$$\frac{\sigma_1\sigma_2 - 3\sigma_3}{2\sigma_2}.$$

If we now take the Euler-line of the triangle to be the axis of reals, σ_1 is real and $\sigma_2 = \sigma_1\sigma_3$. The flex-line meets the Euler-line at the point $\frac{\sigma_1}{2} - \frac{3}{2\sigma_1}$. This gives a simple method of constructing the line of flexes and the points of inflexion, as follows:— Draw the Euler-line through σ_1, the orthocentre and O, the circumcentre of the triangle. Take the circumcentre as the origin, and determine the positive direction of the axis of reals to be from O to σ_1. Construct the point $\frac{\sigma_1}{2} - \frac{3}{2\sigma_1}$ on the Euler-line, and erect a perpendicular to it at this point. This is the line of flexes, and cuts the three sides of the triangle in the three points of inflexion.

Consider again the cubic

$$\begin{cases} x = \dfrac{\sigma_1\sigma_3 - 3t\sigma_3}{\sigma_3 - t^3}, \\ y = -\dfrac{\sigma_2 t^3 - 3\sigma_3 t^2}{\sigma_3(\sigma_3 - t^3)}. \end{cases}$$

As t^3 approaches σ_3, x and y become infinite, hence the cubic has three branches, each running to infinity. Let us examine it for asymptotes. Call $\sqrt[3]{\sigma_3} = \tau$, and take any line

$$x + \lambda y = \text{const.}$$

Let this line cut the cubic at any point t. Then its equation is

$$x + \lambda y = \frac{\sigma_3^2(\sigma_1 - 3t) - \lambda(\sigma_2 t^3 - 3\sigma_3 t^2)}{\sigma_3(\sigma_3 - t^3)}. \tag{13}$$

If $t = \tau$ the left hand member of (13), when cleared of fractions, vanishes, and the right hand member gives

$$\lambda = \frac{\sigma_3^2(\sigma_1 - 3\tau)}{\sigma_2\tau^3 - 3\sigma_3\tau^2}.$$

Putting this value of λ in (13) we have after putting $t = \tau$ and reducing,

$$(\sigma_2\tau - 3\sigma_3)x + \sigma_3\tau(\sigma_1 - 3\tau)y = \sigma_1\sigma_2\tau - 2\sigma_2\tau^2 - 2\sigma_1\sigma_3 + 3\sigma_3\tau. \tag{14}$$

This for the three values of τ gives the three asymptotes.

The line (14) is perpendicular to the line joining the point τ to the centroid, $\sigma_1/3$, of the triangle; for the equation of this line is

$$\begin{vmatrix} x & y & 1 \\ \tau & \frac{1}{\tau} & 1 \\ \frac{\sigma_1}{3} & \frac{\sigma_2}{3\sigma_3} & 1 \end{vmatrix} = 0,$$

or

$$(15) \qquad (\sigma_2\tau - 3\sigma_3)x - \sigma_3\tau(\sigma_1 - 3\tau)y = \sigma_2\tau^2 - \sigma_1\sigma_3.$$

The lines (14) and (15) meet at the point $x = (\sigma_1 - \tau)/2$.

Since τ is a turn this point lies on the Feuerbach-circle of the triangle, whose equation is $x = (\sigma_1 - t)/2$.

Hence the

THEOREM IX. *The cubic has three asymptotes each perpendicular to the line joining the centroid of the triangle to the point τ of the unit circle which gives the infinity of the cubic, and cutting this line at the point $(\sigma_1 - \tau)/2$, a point on the Feuerbach-circle of the triangle.*

From this theorem the construction of the asymptotes is evident.

Since the cubic has three real points of inflexion it has an isolated double point given by the neutral pair, *i. e.* the Hessian, of the involution.

The Hessian of (7) is

$$(16) \qquad (\sigma_1^2 - 3\sigma_2)t^2 + (9\sigma_3 - \sigma_1\sigma_2)t + \sigma_2^2 - 3\sigma_1\sigma_3 = 0.$$

If h and h' be the roots of (16)

$$hh' = \frac{\sigma_2^2 - 3\sigma_1\sigma_3}{\sigma_1^2 - 3\sigma_2},$$

$$h + h' = \frac{\sigma_1\sigma_2 - 9\sigma_3}{\sigma_1^2 - 3\sigma_2}.$$

The double point is

$$x = \frac{(\sigma_1 - 3h)\sigma_3}{\sigma_3 - h^3} = \frac{(\sigma_1 - 3h')\sigma_3}{(\sigma_3 - h'^3)} = \frac{3\sigma_3(h - h')}{h^3 - h'^3},$$

or

$$x = \frac{3\sigma_3}{h^2 + hh' + h'^2},$$

or

$$x = \frac{\sigma_3(\sigma_1^2 - 3\sigma_2)^2}{27\sigma_3^2 - 9\sigma_1\sigma_2\sigma_3 + \sigma_1^3\sigma_3 + \sigma_2^3}\,.$$

5. Further treatment of the cusp-locus. The cusp-locus of section 4 arises also in another way. Consider a deltoid touching three lines. Make it touch a fourth line, L. Determine the point of contact P, of this line and the deltoid. Move the line L parallel to itself, and the point of contact will be proved to run along a line, and the envelope of this line, as the direction of the line L varies, will be the cusp-locus of the preceding section. For from section 2 we have the deltoid touching $\alpha\beta\gamma$ given by

$$(k-1)x + \sigma_1 = \lambda\left(t + t_1 + \frac{1}{tt_1}\right),$$

together with

$$\lambda^3 = \sigma_3 k.$$

Eliminating k from these two, we have

$$(1) \qquad \left(\frac{\lambda^3}{\sigma_3} - 1\right)x + \sigma_1 = \lambda\left(t + t_1 + \frac{1}{t_1 t}\right).$$

Its conjugate is

$$\left(\frac{\sigma_3}{\lambda^3} - 1\right)y + \frac{\sigma_2}{\sigma_3} = \frac{1}{\lambda}\left(\frac{1}{t} + \frac{1}{t_1} + t_1 t\right).$$

Eliminating t from these two we have

$$(2) \qquad \left(\frac{\lambda^3}{\sigma_3} - 1\right)t_1 x - \left(\frac{\sigma_3}{\lambda^3} - 1\right)\lambda^2 y = \frac{\lambda^2\sigma_2}{\sigma_3} - \frac{\lambda}{t_1} - \sigma_1 t_1 - \lambda t_1^2.$$

This is the equation of the tangent at the point t_1. If this be parallel to any line

$$x + ky = b,$$

then k is given by the equation

$$k = \frac{-\left(\frac{\sigma_3}{\lambda^3} - 1\right)\lambda^2}{\left(\frac{\lambda^3}{\sigma_3} - 1\right)t_1} = \frac{\sigma_3}{t_1\lambda}.$$

Hence

$$(3) \qquad t_1 = \frac{\sigma_3}{k\lambda}.$$

The point of contact in general is given by

$$(4)\qquad x\left(\frac{\lambda^3}{\sigma_3}-1\right)+\sigma_1=\lambda\left(2t_1+\frac{1}{t_1^2}\right).$$

From (3) and (4) we have

$$(5)\qquad x\left(\frac{\lambda_3}{\sigma_3}-1\right)+\sigma_1=\frac{2\sigma_3}{k}+\frac{k^2\lambda^3}{\sigma_3^2}\,.$$

Let $\frac{1}{k}=-a$, and let the Euler-line be the axis of reals. Then (5) and its conjugate may be written in the form

$$\begin{cases} x\left(\dfrac{\lambda^3}{\sigma_3}-1\right)+\sigma_1=-2a\sigma_3+\dfrac{\lambda^3}{a^2\sigma_3^2}, \\ y\left(\dfrac{\sigma^3}{\lambda^3}-1\right)+\sigma_1=-\dfrac{2}{a\sigma_3}+\dfrac{a^2\sigma_3^2}{\lambda^3}\,. \end{cases}$$

Eliminate λ from these two, and we find the locus of P to be

$$(6)\quad \left(\sigma_1+\frac{2}{a\sigma_3}-a^2\sigma_3\right)x+\left(\sigma_1+2a\sigma_3-\frac{1}{a^2\sigma_3}\right)y=3+2a\sigma_1\sigma_3+\frac{2\sigma_1}{a\sigma_3}+\sigma_1^2\,,$$

which is a line. Hence the

THEOREM X. *If the tangents be drawn in a given direction to the deltoids inscribed to a given triangle, the locus of the point of tangency is a line.*

To obtain the envelope of this line, we differentiate (6) with regard to a and we have

$$(7)\qquad \left(\frac{1}{a\sigma_3}+a^2\sigma_3\right)x-\left(a\sigma_3+\frac{1}{a^2\sigma_3}\right)y=-\sigma_1\left(a\sigma_3-\frac{1}{a\sigma_3}\right).$$

This together with (6) gives the envelope of the line. To obtain the map-equation of this curve we eliminate y from (7) and (6) and obtain

$$x=\frac{\sigma_1+3a\sigma_3}{1+a^3\sigma_3^2}.$$

But since a and σ_3 are both turns we may write $-a\sigma_3=t$. Then this equation becomes

$$x=\frac{\sigma_1-3t}{1-\dfrac{t^3}{\sigma_3}}\,.$$

Hence we have the

Theorem XI. *The envelope of the line of Theorem X is the cusp-locus of the preceding section.*

6. The theory of osculants. Study's theory of osculants is briefly indicated in the *Leipziger Berichte.** We shall work it out here for the present rational point cubic from the standpoint of the geometry of strokes, but the results are all a part of Study's theory.

Consider again the cubic

$$x = \frac{\sigma_1 - 3t}{1 - \dfrac{t^3}{\sigma_3}} .$$

For each t there is one x, but for each x there are three t's. Let them be t_1, t_2, t_3 and consider the equation

$$x = \frac{\sigma_1 - (t_1 + t_2 + t_3)}{1 - \dfrac{t_1 t_2 t_3}{\sigma_3}} , \tag{1}$$

from which the cubic is obtained by making the three t's equal.

Let t_3 vary, *i. e.*, put $t_3 = t$, and we get

$$x = \frac{[\sigma_1 - (t_1 + t_2 + t)]\sigma_3}{\sigma_3 - t_1 t_2 t} . \tag{2}$$

This is a line, for any expression for x which is of the first degree in t gives a circle, but for $t = \dfrac{\sigma_3}{t_1\, t_2}$ x becomes infinite, hence the circle becomes a line. What line this is we shall see later (page 132).

In (2) put $t_1 = t_2$ and we have

$$x = \frac{[\sigma_1 - (2t_1 + t)]\sigma_3}{\sigma_3 - t_1^2 t} . \tag{3}$$

For a given t_1 this is a line, for a given t, and varying t_1 it is a conic (see page 131).

Hold t_1 fixed and consider the line (3). It is the tangent to the cubic at the point t_1, for

$$D_t x]_{t = t_1} = \frac{3\sigma_1 t_1^2 - 6t_1^3 - 3\sigma_3}{(\sigma_3 - t_1^3)^2}\, \sigma_3, \text{for the cubic,}$$

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* E. Study: Ueber die Raumcurven vierter Ordnung, zweiter Art, *Leipziger Berichte*, 1886.

and

$$D_t x\big]_{t=t_1} = \frac{\sigma_1 t_1^2 - 2t_1^3 - \sigma_3}{(\sigma_3 - t_1^3)^2}\,\sigma_3, \text{ for the line.}$$

I shall show this directly in order to make use of some of the results so obtained for further investigation. The conjugate of (3) is

$$(3') \qquad y = \frac{\sigma_3 t_1^2 + 2\sigma_3 t_1 t - \sigma_2 t_1^2 t}{\sigma_3(\sigma_3 - t_1^2 t)}\,.$$

Eliminating t from (3) and (3′) we have the equation of the line in the form

$$(4) \qquad \frac{t_1^4 - \sigma_2 t_1^2 + 2\sigma_3 t_1}{\sigma_1\sigma_2 t_1^2 + 3\sigma_3 t_1^2 - 2\sigma_1\sigma_3 t_1 - 2\sigma_2 t_1^3}\,x + \frac{\sigma_3(\sigma_3 - \sigma_1 t_1^2 + 2t_1^3)}{\sigma_1\sigma_2 t_1^2 + 3\sigma_3 t_1^2 - 2\sigma_1\sigma_3 t_1 - 2\sigma_2 t_1^3}\,y + 1 = 0$$

This line cuts the cubic in three points given by the equation

$$(5) \qquad t^3 - \frac{3(\sigma_3 + 2t_1^3 - \sigma_1 t_1^2)}{3t_1^2 - 2\sigma_1 t_1 + \sigma_2}\,t^2 + \frac{3(t_1^4 + 2\sigma_3 t_1 - \sigma_2 t_1^2)}{3t_1^2 - 2\sigma_1 t_1 + \sigma_2}\,t - \frac{t_1^2(\sigma_1 t_1^2 - 2\sigma_2 t_1 + 3\sigma_3)}{3t_1^2 - 2\sigma_1 t_1 + \sigma_2} = 0.$$

But the roots of this equation are

$$t_1,\ t_1, \text{ and } \frac{\sigma_1 t_1^2 - 2\sigma_2 t_1 + 3\sigma_3}{3t_1^2 - 2\sigma_1 t_1 + \sigma_2}\,.$$

Hence the line (3) is tangent to the cubic at the point t_1.

The results of section 4 may be readily obtained from this point of view. The points of inflexion are obtained by making the roots of (5) equal, *i. e.*, by putting

$$t_1 = \frac{\sigma_1 t_1^2 - 2\sigma_2 t_1 + 3\sigma_3}{3t_1^2 - 2\sigma_1 t_1 + \sigma_2}\,,$$

or

$$t_1^3 - \sigma_1 t_1^2 + \sigma_2 t - \sigma_3 = 0.$$

This gives the points of inflexion as before to be α, β, and γ. Putting these values of t_1 in (3) we have the map-equations of the flex-tangents:

$$(6)\qquad \begin{cases} x = \dfrac{\sigma_1\sigma_3 - 2\alpha\sigma_3 - t\sigma_3}{\sigma_3 - \alpha^2 t}, \\[2ex] x = \dfrac{\sigma_1\sigma_3 - 2\beta\sigma_3 - t\sigma_3}{\sigma_3 - \beta^2 t}, \\[2ex] x = \dfrac{\sigma_1\sigma_3 - 2\gamma\sigma_3 - t\sigma_3}{\sigma_3 - \gamma^2 t}. \end{cases}$$

The asymptotes are obtained by putting $t_1 = \tau$ in (3), and are for the three values of τ

$$(7)\qquad x = \frac{\sigma_1\sigma_3 - 2\tau\sigma_3 - t\sigma_3}{\sigma_3 - \tau^2 t}.$$

Equations (6) and (7) are the map-equations corresponding to equations (9) and (14), respectively, of section 4, as may be readily verified by writing their conjugates and eliminating t.

Consider again equation (3). Hold t fixed and let t_1 vary, or what is the same thing let t vary when the equation is written in the form

$$(8)\qquad x = \frac{(\sigma_1 - 2t - t_1)\sigma_3}{\sigma_3 - t_1 t^2}.$$

Equation (8) is evidently a conic, for the involution of points in which it cuts any line is a quadratic involution. This conic has the point $t = t_1$ in common with the cubic, and is tangent to the cubic at that point, for

$$D_t x\big]_{t=t_1} = \frac{3\sigma_1 t_1^2 - 6t_1^3 - 3\sigma_3}{(\sigma_3 - t_1^3)^2}\,\sigma_3, \text{ for the cubic}$$

and

$$D_t x\big]_{t=t_1} = \frac{2\sigma_1 t_1^2 - 4t_1^3 - 2\sigma_3}{(\sigma_3 - t_1^3)^2}\,\sigma_3, \text{ for the conic.}$$

The conic (8) is called the *osculant conic* or *first osculant* of the cubic at the point t_1, and the line (3) is called the *second osculant* at that point. For each t_1, equation (8) gives a conic, hence for a variable t_1 we obtain a whole system of *osculant conics*.

The conic (8) touches the three sides of the given triangle; for, take any side

$$x + \beta\gamma y = \beta + \gamma.$$

This line cuts the conic at the points given by the roots of the equation

$$[\sigma_3\beta\gamma - \sigma_2\beta\gamma t_1 + \sigma_3(\beta + \gamma)t_1]t^2 - 2(\sigma_3^2 - \sigma_3\beta\gamma t_1)t + \sigma_1\sigma_3^2 - \sigma_3^2 t_1 - \sigma_3^2(\beta + \gamma) = 0,$$

or

$$(9) \qquad t^2 - 2at + a^2 = 0.$$

Hence the conic touches the line at the point $t = a$. In like manner we can show that the conic (8) touches the other two sides of the triangle. But since equation (9) is independent of t_1, every conic of the system touches the three sides of the triangle. Hence we have the

THEOREM XII. *The system of the osculant conics of the cubic of section 4 are inscribed to the triangle of the flex-tangents of the cubic.*

Consider any two conics of the system:

$$(a) \qquad x = \frac{(\sigma_1 - 2t - t_1)\sigma_3}{\sigma_3 - t_1 t^2},$$

and

$$(b) \qquad x = \frac{(\sigma_1 - 2t - t_2)\sigma_3}{\sigma_3 - t_2 t^2}.$$

These two conics have four common tangents. Three of them are the sides of the triangle and the fourth is the line

$$(2) \qquad x = \frac{(\sigma_1 - t_1 - t_2 - t)\sigma_3}{\sigma_3 - t_1 t_2 t};$$

for this line (2) has the point t_2 in common with the conic (a), and since

$$D_t x]_{t=t_2} = \frac{2\sigma_3(\sigma_1 t_1 t_2 - t_1^2 t_2 - t_1 t_2^2 - \sigma_3)}{(\sigma_3 - t_1 t_2^2)^2}, \text{ for the conic,}$$

and

$$D_t x]_{t=t_2} = \frac{\sigma_3(\sigma_1 t_1 t_2 - t_1^2 t_2 - t_1 t_2^2 - \sigma_3)}{(\sigma_3 - t_1 t_2^2)^2}, \text{ for the line,}$$

it is tangent to the conic (a). In like manner we can show that it is also tangent to the conic (b); hence it is their common tangent.

Consider with (a) and (b) also a third conic of the system,

$$(c) \qquad x = \frac{(\sigma_1 - 2t - t_3)\sigma_3}{(\sigma_3 - t_3 t^2)}.$$

The fourth common tangent of (*a*) and (*b*) is

$$x = \frac{(\sigma_1 - t_1 - t_2 - t)\,\sigma_3}{\sigma_3 - t_1 t_2 t},$$

that of (*b*) and (*c*) is

$$x = \frac{(\sigma_1 - t_2 - t_3 - t)\sigma_3}{\sigma_3 - t_2 t_3 t}.$$

and that of (*c*) and (*a*) is

$$x = \frac{(\sigma_1 - t_3 - t_1 - t)\sigma_3}{\sigma_3 - t_3 t_1 t}.$$

These three lines meet at the point

$$x = \frac{(\sigma_1 - t_1 - t_2 - t_3)\sigma_3}{\sigma_3 - t_1 t_2 t_3}. \tag{1}$$

Let us consider the conics of the system in this way. Any conic of the system may be written in the form

$$x = \frac{(\sigma_1 - 2t - k)\,\sigma_3}{\sigma_3 - kt^2}, \tag{10}$$

where k is a constant in absolute value unity. A tangent at the point t_1 is

$$x = \frac{(\sigma_1 - t_1 - t - k)\sigma_3}{\sigma_3 - kt_1 t}, \tag{11}$$

another tangent at the point t_2 is

$$x = \frac{(\sigma_1 - t_2 - t - k)\sigma_3}{\sigma_3 - kt_2 t}. \tag{12}$$

These two tangents meet at the point

$$x = \frac{(\sigma_1 - t_1 - t_2 - k)\sigma_3}{\sigma_3 - t_1 t_2 k}. \tag{13}$$

7. The system of osculant conics. We can use the formulæ obtained in the latter part of the preceding section to prove an interesting theorem with regard to the system of osculant conics of the cubic of section 4. The points at infinity on the conic (10), section 4, are given by $t^2 = \frac{\sigma_3}{k}$. Hence from (11) and (12) of the same section, the asymptotes are

$$x = \frac{\left(\sigma_1 - \sqrt{\frac{\sigma_3}{k}} - t - k\right)\sigma_3}{\sigma_3 - \sqrt{\frac{\sigma_3}{k}}\,kt},$$

and

$$x = \frac{\left(\sigma_1 + \sqrt{\frac{\sigma_3}{k}} - t - k\right)\sigma_3}{\sigma_3 + \sqrt{\frac{\sigma_3}{k}}\,kt}\,.$$

Their point of intersection is given by putting $t_1 = \sqrt{\frac{\sigma_3}{k}}$, and $t_2 = -\sqrt{\frac{\sigma_3}{k}}$ in equation (13). The result thus obtained is

$$(1) \qquad x = \frac{\sigma_1 - k}{2}\,.$$

As k runs around the unit circle this is the locus of centres of the system of osculant conics, but (1) is the map-equation of the Feuerbach-circle of the triangle; hence the

THEOREM XIII. *The system of conics touching three lines and having their centres on the Feuerbach-circle of the three lines, touches the cubic cusp-locus of the system of deltoids inscribed to the three-line, (i. e. the system of deltoids which are the envelopes of the asymptotes of a system of similar conics circumscribed to the triangle of the three lines).*

8. The envelope of a pair of conjugate diameters of a system of similar conics through three points. Equations (11) and (12) of section 3 are the asymptotes of a system of such conics passing through the three points α, β, and γ. Write these equations in the forms

$$(1) \quad \equiv \frac{\sigma_3(k^2-1)}{a}\,x - \sigma_3 k(k^2-1)y - ak^2 - \sigma_1\sigma_3 k - \frac{\sigma_1\sigma_3 k^2}{a} - \frac{\sigma_3^2 k}{a^2} = 0,$$

and

$$(2) \quad \equiv -\frac{k\sigma_3(k^2-1)}{a}\,x + \sigma_3(k^2-1)\,y - ak - \sigma_1\sigma_3 k^2 - \frac{\sigma_1\sigma_3 k}{a} - \frac{\sigma_3^2 k^2}{a^2} = 0.$$

Any line $(1) + \lambda(2) = 0$, (where λ is real) or,

$$(3) \qquad \frac{\sigma_3(k^2-1)(1-\lambda k)x}{a} + \sigma_3(k^2-1)(\lambda-k)y - ak(\lambda+k)$$
$$- \sigma_1\sigma_3 k(1+\lambda k) - \frac{\sigma_1\sigma_3 k(\lambda+k)}{a} - \frac{\sigma_3^2 k(1+\lambda k)}{a^2} = 0,$$

is a diameter, making with the line (1) a constant angle ϕ, where ϕ is given by the equation

$$e^{2i\phi} = \frac{\lambda - k}{k(1 - \lambda k)} \, .$$

To obtain the envelope of this line, we differentiate (3) with regard to a and obtain

$$-\frac{\sigma_3(k^2 - 1)(1 - \lambda k)x}{a^2} - k(k + \lambda) + \frac{\sigma_1 \sigma_3 k(k + \lambda)}{a^2} + \frac{2\sigma_3^2 k(1 + \lambda k)}{a^3} = 0.$$

Solving for x and putting $\frac{\sigma_3}{a} = t$, we have

$$(4) \qquad x = \frac{\sigma_1 k(k + \lambda)}{(k^2 - 1)(1 - \lambda k)} + \frac{2k(1 + \lambda k)t}{(k^2 - 1)(1 - \lambda k)} - \frac{\sigma_3 k(k + \lambda)}{(k^2 - 1)(1 - \lambda k)t^2} \, ,$$

a deltoid. Hence we have the

THEOREM XIV. *The envelope of any diameter making a fixed angle with the asymptotes of a system of similar conics through three points is a deltoid.*

Let us consider the pairs of conjugate diameters. The two lines

$$(1) + \lambda(2) = 0$$

and

$$(1) - \lambda(2) = 0$$

are partners in an involution of which $(1) = 0$ and $(2) = 0$ are the double lines. Hence $(1) + \lambda(2) = 0$ and $(1) - \lambda(2) = 0$ are conjugate diameters of the conic of which (1) and (2) are the asymptotes.*

If we put $-\lambda$ for λ in (4), we have

$$(5) \qquad x = \frac{\sigma_1 k(k - \lambda)}{(k^2 - 1)(1 + \lambda k)} + \frac{2k(1 - \lambda k)t}{(k^2 - 1)(1 + \lambda k)} - \frac{\sigma_3 k(k - \lambda)}{(k^2 - 1)(1 + \lambda k)t^2} \, .$$

The two deltoids (4) and (5) have three common tangents, for, writing the conjugate of (4) and combining it with (4) we have

$$(6) \qquad (k^2 - 1)(1 - \lambda k)t^2 x - \sigma_3(k^2 - 1)(k - \lambda)ty + k(1 + \lambda k)t^3 + \sigma_1 k(k + \lambda)t^2 + \sigma_2 k(1 + \lambda k)t + \sigma_3 k(k + \lambda) = 0.$$

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* See Salmon's *Conic Sections*, p. 296.

This for each t is a tangent of (4). Doing the same thing for (5) for some other t, say t_1, we have

$$(7)\qquad (k^2-1)(1+\lambda k)t_1^2x-\sigma_3(k^2-1)(k+\lambda)t_1y+k(1-\lambda k)t_1^3 + \sigma_1k(k-\lambda)t_1^2+\sigma_2k(1-\lambda k)t_1+\sigma_3k(k-\lambda)=0.$$

Put $t_1=\mu t$ in (7) and we have

$$(8)\qquad (k^2-1)(1+\lambda k)\mu^2t^2x-\sigma_3(k^2-1)(k+\lambda)\mu ty+k(1-\lambda k)\mu^3t^3 + \sigma_1k(k-\lambda)\mu^2t^2+\sigma_2k(1-\lambda k)\mu t+\sigma_3k(k-\lambda)=0.$$

Make (6) and (8) identical and we have

$$(9)\qquad \mu=\frac{(1-\lambda k)(k+\lambda)}{(k-\lambda)(1+\lambda k)},$$

and

$$(10)\qquad [(1+\lambda k)-\mu^3(1-\lambda k)]t^3+\sigma_1[(k+\lambda)-(k-\lambda)\mu^2]t^2 + \sigma_2[(1+\lambda k)-(1-\lambda k)\mu]t+\sigma_3[(k+\lambda)-(k-\lambda)]=0.$$

Eliminating μ from (9) and (10) we obtain

$$(11)\qquad t^3+\sigma_1At^2+\sigma_2Bt+\sigma_3C=0,$$

where

$$\begin{cases} A=\dfrac{(k+\lambda)(k-\lambda)^3(1+\lambda k)^3-(k+\lambda)^2(k-\lambda)^2(1+\lambda k)(1-\lambda k)^2}{(1+\lambda k)^4(k-\lambda)^3-(1-\lambda k)^4(k+\lambda)^3}, \\ B=\dfrac{(1+\lambda k)^4(k-\lambda)^3-(1-\lambda k)^2(k+\lambda)(1+\lambda k)^2(k-\lambda)^2}{(1+\lambda k)^4(k-\lambda)^3-(1-\lambda k)^4(k+\lambda)^3}, \\ C=\dfrac{(k+\lambda)(k-\lambda)^3(1+\lambda k)^3-(k-\lambda)^4(1+\lambda k)^3}{(1+\lambda k)^4(k-\lambda)^3-(1-\lambda k)^4(k+\lambda)^3}. \end{cases}$$

Since (11) is a cubic in t, there are three lines in general touching (4) and (5), given by putting the roots of (11) in (6).

As k varies the cusp locus of the system of deltoids enveloped by the two conjugate diameters is obtained in the same way as in section 2, and is found to be

$$(12)\qquad x=\frac{\sigma_1(k+\lambda)-3\sqrt[3]{\sigma_3(1+\lambda k)^2(k+\lambda)}}{(k^2-1)(1-\lambda k)}\,k.$$

And the locus of their centres is a cubic curve given by

$$(13)\qquad x=\frac{\sigma_1k(k+\lambda)}{(k^2-1)(1-\lambda k)}.$$

If we put $\lambda = 1$, the conjugate diameters bisect the angles between the asymptotes and become the axes of the conic, and we have the theorem proved by Professor Allardice* in the January number of the *Transactions of the American Mathematical Society*. For, putting $\lambda = +1$, in (4) and (5) we have the two deltoids, considered by Professor Allardice, in the forms

$$x = -\frac{\sigma_1 k}{(k-1)^2} - \frac{2kt}{(k-1)^2} + \frac{\sigma_3 k}{(k-1)^2 t^2}, \tag{14}$$

and

$$x = \frac{\sigma_1 k}{(k+1)^2} + \frac{2kt}{(k+1)^2} - \frac{\sigma_3 k}{(k+1)^2 t^2}. \tag{15}$$

Equation (11), for $\lambda = 1$, becomes

$$t^3 + \sigma_1 t^2 + \sigma_2 t + \sigma_3 = 0, \tag{16}$$

and equation (6) becomes

$$(k-1)^2 t^2 x + \sigma_3 (k-1)^2 t y - k(t^3 + \sigma_1 t^2 + \sigma_2 t + \sigma_3) = 0. \tag{17}$$

The roots of (16) are $-a$, $-\beta$, $-\gamma$, and putting these values of t in (17) we have

$$\begin{cases} x - \beta\gamma y = 0, \\ x - \gamma a y = 0, \\ x - a\beta y = 0, \end{cases} \tag{18}$$

the three common lines of (14) and (15). But equations (18) are the three perpendicular bisectors of the sides of the given triangle, hence we have the theorem of Professor Allardice,—*The envelope of axes of a system of similar conics consists of two deltoids touching three concurrent straight lines.* Now since equations (18) are independent of k, every such pair of deltoids touches these three lines. Hence as k varies, equations (14) and (15) give a system of deltoids all touching the three concurrent lines. Their cusps run along three other concurrent lines, for, putting $\lambda = 1$ in (12), we obtain

$$x = \frac{-(\sigma_1 - 3\sqrt[3]{\sigma_3})k}{(1-k)^2},$$

and eliminating k from this and its conjugate we have

$$(\sigma_2 - 3\sqrt[3]{\sigma_3^2})x + (\sigma_1 - 3\sqrt[3]{\sigma_3})y = 0. \tag{19}$$

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*R E. Allardice: On the envelope of axes of a system of conics through three fixed points; *Trans. Amer. Math. Soc.*, vol. 4 (1903), p 103.

But this, for the three cube roots of σ_3; gives three lines through the circumcentre of the given triangle. The centres of the deltoids of this system lie on a line; for equation (13), for $\lambda = 1$, becomes

$$x = -\frac{\sigma_1 k}{(1-k)^2}\,.$$

And if we eliminate k from this and its conjugate we obtain

$$\sigma x - \sigma_1 \sigma_3 y = 0. \tag{20}$$

But this is the equation of the Euler-line of the triangle. Hence we have the

THEOREM XV. *The locus of the centres of the system of deltoids touching the perpendicular bisectors of a given triangle, is the Euler-line of that triangle, and the locus of their cusps is three straight lines through the circumcentre of the triangle.*

A particular case of special interest here is that of the parabola.* For $k^2 = 1$, the asymptotes become two coincident lines at infinity, and hence the system of conics reduces to a system of parabolas through the three points. The envelope of the axis of this system of parabolas is thus obtained by putting $k^2 = 1$, in (14) and (15). For $k = 1$, (14) gives nothing, while (15) gives

$$x = \frac{\sigma_1}{4} + \frac{t}{2} = \frac{\sigma_3}{4t^2}, \tag{21}$$

a deltoid with its centre at $\frac{\sigma_1}{4}$ and its size $\frac{1}{4}$. This deltoid is the minimum or Steiner deltoid of the triangle formed by joining the mid-points of the sides of the given triangle, which I shall call for shortness the mid-point triangle of the given triangle. To prove this, call $\alpha'\beta'\gamma'$ the mid-point triangle, where

$$\alpha' = \frac{\beta+\gamma}{2}, \quad \beta' = \frac{\alpha+\gamma}{2}, \quad \gamma' = \frac{\alpha+\beta}{2}\,.$$

Since (21) is one of the deltoids of the system (15) it touches the perpendicular bisectors of $\alpha\beta\gamma$, *i. e.* the perpendiculars of $\alpha'\beta'\gamma'$.

The general bilinear transformation in the plane will carry any three points into any other three points and hence will be determined by three pairs of corresponding points.* Let the general bilinear transformation be

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* Since the above was written Professor Bromwich has called attention to this case. *Cf.* T. J. I'A. Bromwich, Similar conics through three points, *Trans. Amer. Math. Soc.* vol. 4 (1903), p. 489.

$$x_1 = \frac{ax+b}{cx+d}, \tag{22}$$

and let the constants be so determined that it will carry α into α', β into β', and γ into γ'. Then we have

$$a = -1, \quad b = \sigma_1, \quad c = 0, \quad d = 2,$$

and (22) becomes

$$x_1 = \frac{\sigma_1 - x}{2}. \tag{23}$$

This transformation carries the triangle $\alpha\beta\gamma$ into the triangle $\alpha'\beta'\gamma'$, and since it is linear in x and x_1, it changes neither the order nor the class of any curve. It then carries the Steiner deltoid of $\alpha\beta\gamma$ into the Steiner deltoid of $\alpha'\beta'\gamma'$. The Steiner deltoid of $\alpha\beta\gamma$ is from equation (5′) of section 2.

$$x = \frac{\sigma_1}{2} + t + \frac{\sigma_3}{2t^2}. \tag{24}$$

Applying the transformation (23) to this we obtain

$$x_1 = \frac{\sigma_1}{4} - \frac{t}{2} - \frac{\sigma_3}{4t^2}.$$

but for $t = -t$, this is equation (20). Hence equation (20) is the Steiner deltoid of the triangle $\alpha'\beta'\gamma'$. This may be readily verified by direct methods. Hence we have the

THEOREM XVI. *The envelope of the axis of a system of parabolas circumscribed to a given triangle is the minimum deltoid inscribed to the mid-point triangle of the given triangle.*

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MARCH 23, 1903.

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*See Harkness and Morley: *Introduction to the Theory of Analytic Functions*, p. 28.